Steps Toward Space Colonization: Colony Location and Transfer Trajectories

T. A. Heppenheimer*
Center for Space Science, Fountain Valley, Calif.

The problem of optimal location of a space colony is treated by consideration of a baseline transfer trajectory from the mass-catching point at L2 to high Earth orbit. Locations treated include the 2:1, 5:2, 7:3, and 3:1 resonances; nominal candidate orbits are found as periodic orbits in the restricted three-body problem. Optimal colony inclination is estimated via the method of phase equilibria. The candidates are evaluated according to criteria derived from operational considerations, including parametric studies of transfer trajectories from L2, and from long-term numerical integrations via Cowell's method in a restricted four-body problem. The 7:3 resonance is eliminated because it is unstable. The 3:1 fails because of excessive ΔV requirements for the transfer. The 5:2 is rejected because it gives difficulties involving a phasing maneuver, required prior to terminal rendezvous. The 2:1 resonance is selected and its advantages are noted.

Introduction

THE concept of colonization of space[1,2] represents a major departure in astronautics, in that it is considered to rest upon the large-scale availability of lunar resources for use in industrial activities. Chief among these are the construction of solar power satellites, which are to be subsequently transported to geosynchronous orbit.

Such a proposal clearly requires understanding of the problem of transport of lunar resources to the colony or manufacturing center. The transport mode treated here involves launch of unprocessed lunar material, by mass-driver or electromagnetic catapult, to be caught in space and subsequently transported to a colony in high Earth orbit. All processing and manufacturing then are done at the colony; lunar operations are restricted to mining and launching of material. The rationale of this choice of mode (as opposed, for example, to putting major industrial operations on the Moon itself) is discussed elsewhere.[2,3] The flight mechanics of lunar-launched material are also discussed elsewhere.[4-6] This paper is concerned with the choice of colony location.

A suitable point of departure is the identification of L2, the translunar libration point, as the catching-point for lunar-launched material.[4] Hence the colony location problem involves transfer trajectories originating from L2. As in Refs. 4-6, the following normalized units are used: unit mass $=(\text{Earth}+\text{Moon})$, lunar mass $=\mu=0.01215$, unit distance $=384{,}410$ km, unit time $=104.362$ h, unit velocity $=1023.17$ m/s, unit acceleration $=0.00273$ m/s^2.

Figure 1 shows the trajectory of a body departing L2 with a relative velocity -0.01, assuming motion in the restricted three-body problem. The plot is in a rotating coordinate system fixed to the Earth-Moon line of centers. Reference 4 notes that the most advantageous time for transfer from this trajectory to a stable colony location is at $t=16.25$ (L2 departure time $=0$) as this corresponds to a peak in eccentricity ($e=0.3681$ at semimajor axis $a=0.5621$), which minimizes perigee $[Q=a(1-e)]$. The lowest-cost transfer (i.e., lowest ΔV) is to a stable orbit with the same Q. A 2:1 resonant orbit is proposed and treated via the method of phase equilibria.[7] This orbit is studied below using numerical integration. Moreover, the cited transfer point is close to not only the 2:1 resonance but also to the 7:3, 5:2, and 3:1 resonances, which thus represent alternate colony location candidates.

The colony must avoid close lunar approaches that produce major perturbations on its orbit. The Goldreich-Greenberg stabilization mechanism[4] prevents such close approaches if the colony is on a resonant orbit. Hence, in this study, candidate colony locations in high Earth orbit are restricted to resonant orbits.

Restricted Three-Body Solution

To treat the problem of colony location, one first develops reference colony orbits in the restricted three-body problem. Such orbits then are studied in a restricted four-body problem, wherein the motions of Earth, Moon, and colony are given by numerical integration. The term "restricted" refers to a representation of the colony as a massless test body; no other approximation is implied by this term.

The numerically computed reference orbits are rendered simple by requiring them to be periodic, with their stability being treated by the linear Floquet theory. In Moon-centered coordinates (x,y) with the negative x axis in the Earth

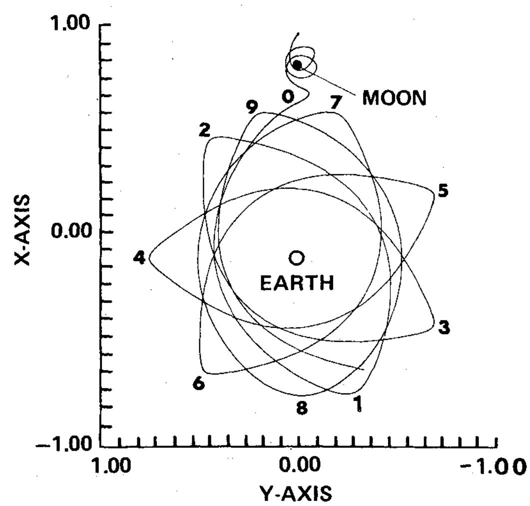

Fig. 1 Transfer trajectory from L2 to the vicinity of the colony. The orbit departs L2 with velocity $\dot{x}=-0.01$. Apogees are numbered in sequence. (Source: Ref. 4)

Received Aug. 8, 1977; revision received Feb. 17, 1978.

Index categories: Earth-Orbital Trajectories; Lunar and Interplanetary Trajectories; Space Station Systems, Manned.

*Chief Scientist. Member AIAA.

direction (L2 is at $x=0.167833, y=0$),

$$\ddot{x}-2\dot{y}=\Omega_x \qquad \ddot{y}+2\dot{x}=\Omega_y$$

$$\Omega=\tfrac{1}{2}[(x+1-\mu)^2+y^2]+(1-\mu)/r_1+\mu/r_2$$

$$r_1^2=(x+1)^2+y^2 \qquad r_2^2=x^2+y^2 \tag{1}$$

One defines a two-point boundary-value problem with initial and final conditions given, the final time t_f being unconstrained

$$\text{at } t=0\text{: } x(0)=x_0 \quad \dot{x}(0)=0 \quad y(0)=0 \quad \dot{y}(0)=\dot{y}_0$$

$$\text{at } t=t_f\text{: } x(t_f)=x_f \quad \dot{x}(t_f)=0 \quad y(t_f)=0 \quad \dot{y}(t_f)=\dot{y}_f \tag{2}$$

The orbit is periodic if $x_0=x_f$, $\dot{y}_0=\dot{y}_f$; otherwise, it is described as re-entrant. Such orbits are found, for fixed x_0, via Newton iteration on $\dot{y}_0$ so as to drive $\dot{x}(t_f)$ to zero. Here x_0 is selected to represent perigee with $Q=0.3552$, with the additional condition that at $t=t_f/2$, the colony and Moon be at superior conjunction unless the colony is then at perigee. The resonances studied are of the type $p{:}q$ and $p-q$ is the order of the resonance. Also, $p-q-1$ is the number of crossings of the x axis, with y increasing, made by the colony in the course of a periodic orbit. The integration time step Δt is 0.01 or 0.02; but after having made $p-q-2$ such crossings, the program uses the test $y(y+\dot{y}\Delta t)<0$ to indicate the imminence of $y=0$ and $t=t_f$. When this occurs, $\Delta t=y/\dot{y}$ is used for the last integration step.

The integrations were performed on a CDC 3300 terminal-operated computer in the 36-bit-word mode, which corresponds to carry ten to eleven significant figures in the computation.

The stability of such orbits is studied using Floquet theory, the algorithm used being essentially that of Refs. 8 and 9. The characteristic equation $|C-I\omega|=0$ provides the stability criterion,[8] where I is the unit matrix and C is the 4×4 matrix of terminal values (at t_f) of a fundamental solution set for the linear variational equations. By this criterion, the stability coefficient A is the negative trace of C and the orbit is stable if $0\le A\le 4$. The orbits of Table 1 were found in this manner and the initial conditions reduced to values of e and a.

The 7:3 resonance is seen to be reentrant rather than periodic, whereas the others are periodic to high accuracy; from its associated value of A, the 7:3 orbit also is seen to be quite unstable.

Figure 2 illustrates the computed orbits, in the coordinate system of Fig. 1, and it is seen that the 2:1, 5:2, and 3:1 cases are simple in form and aesthetically pleasing. The 7:3 orbit by contrast bears more than a casual similarity to the orbit of Fig. 1, despite a choice of initial conditions selected to inhibit or prevent close approaches. The parameters of the 2:1 orbit compare to mean elements estimated in Ref. 1 by the method of phase equilibria: $a=0.6305$, $e=0.35625$. The values of Table 1 are osculating elements.

Table 1 Characteristics of periodic colony orbits

Resonance[a]	2:1	7:3	5:2	3:1
x_0	−1.3552	−1.3552	−0.6448	−1.3552
$\dot{y}_0$	−1.6459373	−1.5978292	1.5704117	−1.51147965
t_f	6.2282222	20.6348647	12.6072431	6.21200083
x_f	−1.3551996	−1.3200706	−0.6448017	−1.35519999
$\dot{x}_f$	−0.0000004	0.0000004	−0.0000000	0.00000000
y_f	−0.0000047	−0.0000436	0.0000031	−0.00000006
$\dot{y}_f$	−1.6459391	−1.7718468	1.5704197	−1.51147969
a	0.6341852	0.5807530	0.5461863	0.48546141
e	0.4399113	0.3883803	0.3496724	0.2683495
A	3.616315	−26.8391964	2.4533212	3.81989431

[a] Note: e, a are osculating elements computed from initial conditions using Eqs. (18).

Out-of-Plane Considerations

The foregoing computations are planar; it is necessary to consider effects due to the inclination of the Moon and colony with respect to the ecliptic. The lunar-orbit plane is inclined to the ecliptic by 5.14 deg. Consequently, if the colony is initially in a coplanar orbit, there are different rates of regression of the lines of nodes of the Moon (due to solar perturbations) and of the colony (due to lunisolar perturbations). Although both Moon (hence, L2) and colony maintain nearly constant inclinations on the ecliptic, their orbit planes mutually precess and in time are mutually inclined by up to 10 deg. The associated plane-change ΔV, for lunar material in transit to a resonant Earth-orbit colony, is some 150 m/s. Hence, it is necessary to treat the problem of inclinations.

The optimum colony inclination can, in principle, be determined using a two-point boundary-value problem in a rotating coordinate system, requiring that the colony's nodal regression rate, $d\Omega_c/dt$, equal the lunar rate, -0.0039991645.[10] However, the integrations must proceed not for one or a few lunar revolutions, but for $t_f=1571.12$ or 18.6133 years, the period of regression of the lunar nodes. The associated integration times are lengthy. Thus, this study was not performed as part of the present investigations.

Estimates of optimal colony inclination are available through the method of phase equilibria. Taking i_m and i_c as the inclinations of Moon and colony on the ecliptic ($i_m=0.0897$), e_m and e_c as the eccentricities, Ω_m and Ω_c as the longitudes of the ascending nodes, and $\tilde{\omega}_m$ and $\tilde{\omega}_c$ as the arguments of perigee, the disturbing function for the secular lunisolar effect upon the colony is[10]

$$R_1=n_s^2a_c^2[\tfrac{1}{4}+\tfrac{3}{8}(e_m^2+e_c^2)-\tfrac{3}{8}(i_m^2+i_c^2)]+\mu\{\tfrac{1}{2}b_{1/2}^{(0)}$$
$$+\tfrac{1}{8}a_cb_{3/2}^{(1)}[e_m^2+e_c^2-i_m^2-i_c^2+2i_mi_c\cos(\Omega_c-\Omega_m)]$$
$$-\tfrac{1}{4}a_cb_{3/2}^{(2)}e_me_c\cos(\tilde{\omega}_c-\tilde{\omega}_m)\} \tag{3}$$

where n_s is the solar mean motion, 0.0748; $b_{1/2}^{(0)}$, etc. are Laplace coefficients which are given by the continued product, convergent for $a<1$:

$$\tfrac{1}{2}b_s^{(j)}(a)=\frac{s(s+1)(s+2)\ldots(s+j-1)}{1\cdot2\cdot3\cdot\ldots\cdot j}a^j\times\left(1+\frac{s(s+j)}{(j+1)}a^2\right.$$
$$\cdot\left[1+\frac{(s+1)(s+j+1)}{2(j+2)}a^2\cdot\left\{1+\frac{(s+2)(s+j+2)}{3(j+3)}a^2\ldots\right.\right. \tag{4}$$

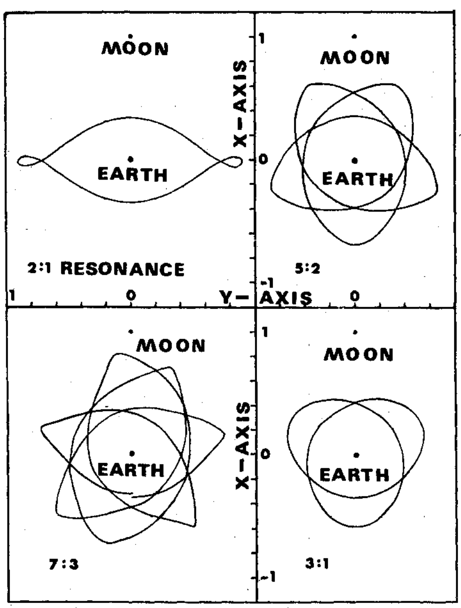

Fig. 2 Candidate colony orbits of Table 1.

Then i_c is found from

$$\frac{d\Omega_c}{dt} = \frac{\partial R_1/\partial i_c}{n_c a_c^2 (1-e_c^2)^{1/2} i_c} = -0.003999164534949 \quad (5)$$

The computation is improved by utilizing the following expansion for the principle part of the solar effect on $d\Omega_c/dt$ (i.e., that part independent of e_c), with $m_c = n_s/n_c$:[10]

$$\frac{1}{n_c}\left(\frac{d\Omega_c}{dt}\right)_s = -0.75m_c^2 + 0.28125m_c^3 + 2.132813m_c^4$$
$$+4.78369m_c^5 + 8.1084m_c^6 + 11.288m_c^7 + \ldots \quad (6)$$

The condition $d\Omega_c/dt = d\Omega_m/dt$ is known as a secular resonance. In dynamical astronomy, such a secular resonance is ordinarily associated with a secular increase in i_c; this obviously is an effect to be avoided here. There is

$$\frac{di_c}{dt} = -\frac{\partial R_1/\partial\Omega_c}{n_c a_c^2 (1-e_c^2)^{1/2} i_c} = \frac{1}{4}\frac{\mu i_m}{n_c a_c (1-e_c^2)^{1/2}} \sin(\Omega_c - \Omega_m) \quad (7)$$

so this effect is avoided for $\Omega_c - \Omega_m = 0$ or π. In the usual theory of secular resonances, one would write $\Omega_c - \Omega_m = (\dot\Omega_c - \dot\Omega_m)t$ and integrate to obtain $i_c = i_c(t)$; the appearance of $(\dot\Omega_c - \dot\Omega_m)$ as a small divisor then constitutes the difficulty noted.

The stability of this solution is found from

$$0 = \begin{vmatrix} \omega - \frac{\partial}{\partial i_c}\left(\frac{di_c}{dt}\right) & -\frac{\partial}{\partial\Omega_c}\left(\frac{di_c}{dt}\right) \\ -\frac{\partial}{\partial i_c}\left(\frac{d\Omega_c}{dt}\right) & \omega - \frac{\partial}{\partial\Omega_c}\left(\frac{d\Omega_c}{dt}\right) \end{vmatrix}$$
$$= \omega^2 + \left(\frac{1}{4}\frac{\mu i_m/i_c}{n_c a_c (1-e_c^2)^{1/2}}\right)^2 \quad (8)$$

where the partial derivatives are evaluated for $\sin(\Omega_c - \Omega_m) = 0$. Hence the solution is stable, and it would be legitimate to write, for example, $(i_c - i_m)^2$ for $[i_c^2 + i_m^2 - 2i_m i_c \cos(\Omega_c - \Omega_m)]$ in Eq. (6). It is seen that the solutions $\Omega_c - \Omega_m = 0, \pi$ are equivalent.

For values of a_c, e_c given in Table 1 and with $n_c^2 a_c^3 = 1$, Table 2 gives the resulting values of $i_c - i_m$, the required constant plane change. For comparison, taking $i_c = i_m$ initially leads eventually to $i_c - i_m = 0.1794$, as noted. Also given in Table 2 are the periods $2\pi/|\omega|$ of free oscillation of (i_c, Ω_c) about the phase equilibria. The associated frequencies ω are found to be close to $(d\Omega_c/dt)_s$ [Eq. (9)], respectively for the 2:1, 5:2, and 7:3 resonances. However, this does not lead to a resonance condition, since there is no coupling between i_c and $(d\Omega_c/dt)_s$.

It is necessary to consider variations in $i_c - i_m$ due to inclination-type resonances. Thus, let λ_m, λ_c be mean longitudes of the Moon and colony. Consider, for example, effects due to the following term of the disturbing function at the 2:1 resonance; this term gives the principal resonant effect on $i_c - i_m$

$$R_2 = \tfrac{1}{8}\mu(i_c - i_m)^2 a_c b_{3/2}^{(3)}(a_c)\cos(4\lambda_m - 2\lambda_c - 2\Omega_c) \quad (9)$$

Table 2 Optimal colony inclinations

Resonance	2:1	7:3	5:2	3:1
$i_c - i_m$, rad	0.01269	0.02408	0.03507	0.08239
$2\pi/\lvert\omega\rvert$, days	11,578.42	13,796.50	15,860.75	23,859.06

For the cited 2:1 orbit, there is a state of phase equilibrium such that $4\lambda_m - 2\lambda_c - 2\tilde\omega_c = 0$; $\tilde\omega_c$ = longitude of perigee. Hence the argument of the cosine in Eq. (9) is $2\tilde\omega_c - 2\Omega_c$ and, while correcting the solar effect on $d\tilde\omega_c/dt$ via Eq. (18) of Ref. 4,

$$\frac{d\tilde\omega_c}{dt} = \frac{(1-e_c^2)^{1/2}}{n_c a_c^2 e_c}\frac{\partial R_1}{\partial e_c} = 0.01366 + 0.00102\sin(\tilde\omega_c - \tilde\omega_m) \quad (10)$$

Then, neglecting the term in $\sin(\tilde\omega_c - \tilde\omega_m)$,

$$\frac{d(i_c - i_m)}{dt} = -\frac{\partial R_2/\partial\Omega_c}{n_c a_c^2 (1-e_c^2)^{1/2} i_c}$$
$$= -0.00701(i_c - i_m)\sin 0.035325t \quad (11)$$

The resonance-induced variation in the required plane change has amplitude 0.1984 times the nominal value of $i_c - i_m$ and period 773.45 days.

Evaluation of the Alternatives

The four alternatives of Table 1 can be evaluated together with two other proposed locations: L5[11] and L2.[12] The evaluation consists of comparison of operational characteristics, including required flight time or ΔV for missions of importance. The proposed L5 location involves a periodic orbit in the restricted four-body problem and in a rotating coordinate system; such an orbit is approximable as a Kepler ellipse of unit semimajor axis and eccentricity 0.1847, around the Earth.[13] Missions of interest include the following:

1) Transfer of lunar material from the transfer point ($a = 0.5621$, $e = 0.3681$), i.e., from the orbit of Fig. 1.
2) Transfer of general cargo from low Earth orbit (LEO; $a = 0.01736$ for LEO at 300 km).
3) Transfer of power satellites to geosynchronous orbit ($a = 0.11024$).
4) Transfer of asteroidal material from a low-C_3 orbit near that of Earth.

The geosynchronous-orbit mission involves what may be the major industrial product of space colonization.[1] The asteroidal mission involves the proposal of O'Leary[14] that Apollo/Amor asteroids, on orbits near to that of Earth (e.g., asteroids 1943, 1976 AA, 1976 UA), represent promising sources of raw materials that can assure long-term colony growth subsequent to an initial phase in which only the Moon is available as an extraterrestrial source of resources.

In Table 3, the transfers are assumed to be Hohmann, with exceptions noted as appropriate, and the greater-than-Keplerian velocity of L2 is taken into account. The subscripts refer to the above-numbered missions.

For L2, ΔV_1 involves transfer from the catcher orbit of Eqs. (4) and (5) of Ref. 1; ΔV_2 and $(t_f)_2$ describe optimal three-impulse transfers involving close lunar fly-bys;[15] ΔV_3 and $(t_f)_3$ are computed assuming the transfer involves the trajectory of Fig. 1 to ($a = 0.5621$, $e = 0.3681$), followed by a Hohmann transfer to geosynchronous orbit. For all cases, ΔV_1 includes plane-change effects as required by Table 2, the plane changes being performed at colony apogee.

This comparison shows that L5 has the characteristics of long trip times and large transfer velocities, particularly from the catcher at L2. It thus is inferior to the alternatives in virtually every respect. This conclusion holds for a catcher at L2. It does not hold if the catcher is at L5; but such a catcher location involves extremely high launch sensitivities.[16] This topic will be treated further in a subsequent communication.

L2, by contrast, has many desirable features. Its apparently overriding disadvantages include its instability. While the technique of stabilization of libration-point satellites has been highly developed,[17] there are obvious disadvantages in being

Table 3 Operational characteristics of proposed colony locations

Site	L5	L2	2:1	7:3	5:2	3:1
a	1.0	1.167833	0.63419	0.58075	0.54619	0.48546
e	0.1847[a]	0.0	0.43991	0.38838	0.34967	0.26832
ΔV_1	0.429[4]	0.01920	0.06117	0.03944	0.04927	0.16976
ΔV_2	3.71661	3.3225[15]	3.60211	3.64994	3.68644	3.76415
$(t_f)_2$	1.46384	1.9754[15]	0.97702	0.83050	0.72797	0.55949
ΔV_3	1.50345	1.2993	1.30104	1.30000	1.29801	1.29072
$(t_f)_3$	1.6367	16.40	1.14997	0.97484	0.86647	0.68703
ΔV_4	0.3586	0.1408	0.35729	0.39336	0.42102	0.48031

[a] Note: The value $e = 0.1847$ for the L5 orbit follows from its appearance, in rotating coordinates, approximately as an L5-centered ellipse of semiminor axis 71,000 km and semimajor axis of very nearly twice this value (see Ref. 13).

required to equip not only the space colony but also all nearby free-flying structures, including construction facilities and power satellites in final assembly, with stationkeeping thrustor systems. Moreover, in the absense of overriding reasons for selecting L2, its unstable nature could prove a source of public concern as to the project's safety. Stationkeeping system failure would lead to the departure of the colony on a trajectory such as Fig. 1, so that retrieving it to L2 would be quite inconvenient. This retrieval would be particularly difficult for a large, heavy colony; and it will be seen that in the absense of such retrieval, the orbit of Fig. 1 might lead to fates rather more unpleasant than merely being on an inconvenient orbit.

Moreover, there are unfavorable tradeoffs involving continuous visibility of Earth, a desirable feature. L2 itself is continuously eclipsed by the Moon. Low-amplitude L2 halo orbits suffer frequent occultation, since their motion is a Lissajous curve in the y-z plane. This is due to the differing frequencies of motion in the y and z directions. "Giant-halo" L2 orbits[15,17] equalize these frequencies and achieve continuous Earth visibility but still are unstable and require $\Delta V_l > 0.1$, thus obviating a major advantage of L2. Recently, Breakwell and Brown[18] have found orbits related to the family of giant-halos and which are stable, in the circular restricted problem. But for these, $\Delta V_l > 0.1$ is typical.

Since in any case an L2 colony would have to be forcibly restrained from following the Fig. 1 trajectory to approach closely the orbits of Fig. 2, it appears preferable to concentrate attention on the latter orbits, to the exclusion of possible libration-point colony locations.

Restricted Four-Body Solution

To gain further insight into the colony orbit candidates, the initial conditions of Table 1 are transformed to rectangular nonrotating Earth-centered coordinates and used in integrations of a planar restricted four-body system. In this system, the motion of the Sun is taken as an unperturbed ellipse with respect to the Earth-Moon barycenter and the lunar motion is given by

$$\ddot{x}_m + \frac{x_m}{r_m^3} = -m_s\left(\frac{x_m - x_s}{r_{ms}^3} + \frac{x_s}{r_s^3}\right) \tag{12a}$$

$$\ddot{y}_m + \frac{y_m}{r_m^3} = -m_s\left(\frac{y_m - y_s}{r_{ms}^3} + \frac{y_s}{r_s^3}\right) \tag{12b}$$

The motion of the colony is given by

$$\ddot{x}_c + \frac{1-\mu}{r^3}x_c = -m_s\left(\frac{x_c - x_s}{r_{cs}^3} + \frac{x_s}{r_s^3}\right) - \mu\left(\frac{x_c - x_m}{r_{cm}^3} + \frac{x_m}{r_m^3}\right) \tag{13a}$$

$$\ddot{y}_c + \frac{1-\mu}{r^3}y_c = -m_s\left(\frac{y_c - y_s}{r_{cs}^3} + \frac{y_s}{r_s^3}\right) - \mu\left(\frac{y_c - y_m}{r_{cm}^3} + \frac{y_m}{r_m^3}\right) \tag{13b}$$

where the subscripts m, s, and c stand for the Moon, Sun, and colony, respectively.

$$\begin{aligned}
r_{ms}^2 &= (x_m - x_s)^2 + (y_m - y_s)^2 && \text{(Sun-Moon distance)}\\
r_m^2 &= x_m^2 + y_m^2 && \text{(Moon-Earth distance)}\\
r_s^2 &= x_s^2 + y_s^2 && \text{(Sun-Earth distance)}\\
r_c^2 &= x_c^2 + y_c^2 && \text{(colony-Earth distance)}\\
r_{cs}^2 &= (x_c - x_s)^2 + (y_c - y_s)^2 && \text{(colony-Sun distance)}\\
r_{cm}^2 &= (x_c - x_m)^2 + (y_c - y_m)^2 && \text{(colony-Moon distance)}
\end{aligned} \tag{14}$$

The following constants are employed:

Solar mass $= m_s = 329{,}426.3$

Solar semimajor axis $= a_s = 389.0548$

Solar eccentricity $= e_s = 0.0168$

Solar mean motion $= n_s = 0.0748013$

The Sun is taken as lying initially at perihelion on the positive x axis with its true anomaly f_s defined by the equation of the center[10]

$$f_s = n_s t + (2e_s - \tfrac{1}{4}e_s^3)\sin n_s t + (5/4)e_s^2 \sin 2n_s t + (13/12)e_s^3 \sin 3n_s t \tag{15}$$

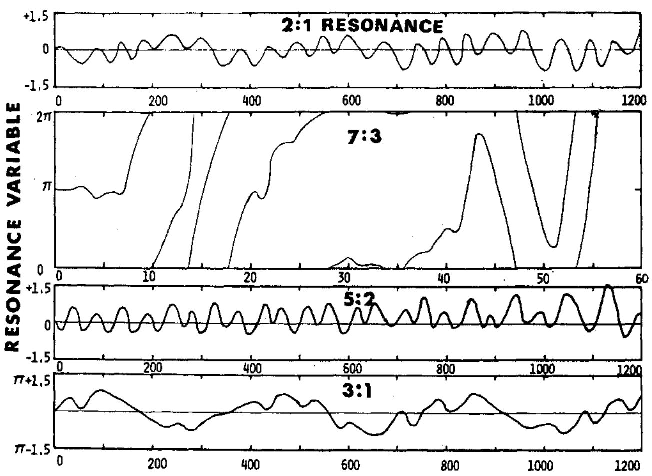

Fig. 3 Behavior of the resonance variable.

so that its coordinates are given by

$$x_s = \mu x_m + r_s \cos f_s \quad (16a)$$

$$y_s = \mu y_m + r_s \sin f_s \quad (16b)$$

$$r_s = a_s (1 - e_s^2)/(1 + e_s \cos f_s) \quad (16c)$$

Lunar motion is initiated also on the positive x axis, a condition of conjunction that in reality leads to the maximum of e_m. Hence lunar motion is initiated at perigee with $a_m = 1.0$, $e_m = 0.0666$, or

$$(x_m)_0 = 0.9334 \qquad (\dot{y}_m)_0 = 1.062459523 \quad (17)$$

and with $(\dot{x}_m)_0 = (y_m)_0 = 0$. The colony state $(x_c, y_c, \dot{x}_c, \dot{y}_c)$ is transformed to Kepler elements via the following equations[10]

$$r_c{}^2 = x_c{}^2 + y_c{}^2 \qquad V_c{}^2 = (\dot{x}_c{}^2 + \dot{y}_c{}^2)/(1-\mu)$$

$$r_c \dot{r}_c = (x_c \dot{x}_c + y_c \dot{y}_c)/(1-\mu)^{1/2}$$

$$a_c = (2/r_c - V_c{}^2)^{-1}$$

$$e_c \sin u_c = a_c{}^{-1/2} r_c \dot{r}_c \qquad e_c \cos u_c = r_c V_c{}^2 - 1$$

$$l_c = u_c - e_c \sin u_c$$

$$\tilde{\omega}_c = \tan^{-1}(y_c/x_c) - 2\tan^{-1}\{[(1+e_c)/(1-e_c)]^{1/2} \tan u_c/2\} \quad (18)$$

with a similar expression for the Moon except that $(1-\mu)$ is replaced by unity. Here u_c is the eccentric anomaly, l_c is the mean anomaly, and for a $p{:}q$ resonance the resonance variable is ϕ

$$\phi = p(l_m + \tilde{\omega}_m) - q(l_c + \tilde{\omega}_c) - (p-q)\tilde{\omega}_c \quad (19)$$

These equations have been integrated for 1200 time units or some 14.26 years using a fifth-order Runge-Kutta integrator and with time step 0.05. The Brouwer error formula[10] then gives an estimate for the expected error in the mean longitude

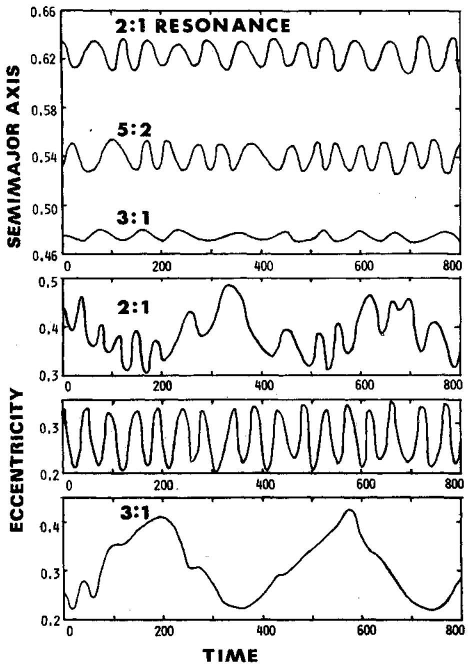

Fig. 4 Behavior of a_c and e_c. The middle plot of e_c refers to the 5:2 resonance.

as 0.00042 rad, this being the most sensitive element. Results of the computations are given in Figs. 3 and 4.

Figure 3 shows the variations in ϕ for the four cases, the time scale being expanded for the 7:3 resonance. Whereas the theory of phase equilibria would predict zero variation in ϕ for the simple cases for which it is applicable, for the three stable resonances ϕ is seen to librate with amplitude ~ 0.3 rad. Three principal libration frequencies, associated with libration amplitudes of this order of magnitude, can be discerned in the computed data: short-period, one month; librational, one-half year; long-period, five years. The plotted data of Fig. 3 have been slightly smoothed by deleting short-period variations. The 2:1 resonance clearly indicates librational and long-period effects, but the 3:1 shows only minor librational variations, the bulk of the plotted variation being long-period.

The instability of the 7:3 resonance manifests itself in wild swings and rapid variations in ϕ, which result from close lunar approaches. At $t = 55.2$, the trajectory is captured into lunar orbit. Hence the 7:3 resonance is not a useful colony location and will not receive further attention.

Figure 4 gives the associated variations in a_c, e_c for the 2:1, 5:2, and 3:1 resonances, short-period variations again being deleted. The comparative characteristics found here, for the 7:3 and 5:2 resonances, echo those found in studies in these same resonances in a planar elliptic Sun-Jupiter system.[19] In Ref. 19, it is also found that the 5:2 resonance gives the possibility of stable libration of ϕ, but there is no libration of ϕ at the 7:3 resonance. In Figs. 3 and 4, it is seen that the long-period variation in ϕ is associated with the largest changes in e_c. On the other hand, the short-period and librational variations produce the largest changes in a_c. Indeed, for the 3:1 case, much of the variation in a_c is short-period, the associated amplitudes being some 0.003 for the short-period, 0.002 for the librational; the plotted curve shows the largest peak-to-peak variations. It must be stated that for different solar or lunar initial conditions, the curves for the different resonances will behave differently; the cited behavior is not necessarily intrinsic to the resonances.

Hence, in general, the perigee Q_c of the colony cannot coincide with the minimum perigee in, for example, Fig. 1. This holds for all three resonances studied. In particular, it is to be noted that Figs. 3 and 4 fail to show any redeeming feature for the 3:1 resonance which can overcome its high ΔV_I as noted in Table 3. Hence the 3:1 option can be eliminated and the surviving candidates are the 5:2 and 2:1 resonances. It now will be seen that a selection between these can be made on the basis of operational aspects of Fig. 1-type transfer trajectories.

Transfer Trajectories

Consider the "artificial-asteroid" method of transport, suggested in Ref. 1 – that the material caught at L2 be consolidated into a single compact mass and that the resulting cargo be equipped with a radio beacon. It then follows a transfer trajectory and the problem is to be able to retrieve it, e.g. with a colony-based tug, with minimum ΔV_I.

To further study such transfer trajectories, Eqs. (12-18) serve to define a circular restricted three-body problem by taking solar mass $m_s = 0$ and, in Eq. (17), $(x_m)_0 = (\dot{y}_m)_0 = 1.0$. The selected time step is $\Delta t = 0.01$ for $r_{cm} > 0.1$, $\Delta t = 0.005$ for $0.1 \geq r_{cm} \geq 0.025$, $\Delta t = 0.0001$ for $r_{cm} > 0.025$. In all cases, motion of the cargo is initiated at L2 with $(x_c)_0 = (\dot{y}_c)_0 = 1.167833$ and $(\dot{x}_c)_0$ subject to choice, the colony variables x_c, y_c, etc., being redesignated as defining motion of this cargo. Here $(\dot{x}_c)_0$ is the departure velocity from L2. The computation of orbital elements by Eqs. (18) is suppressed for $r_{cm} < 0.2$, approximately the lunar sphere of influence; for $r_{cm} \geq 0.2$, they are computed and printed out every 20 integration steps. As an initial test, $(\dot{x}_c)_0 = -0.01$ was taken so as to seek to reproduce Fig. 1.

Table 4 Transfer trajectories – conditions at eccentricity maximum

$(\dot{x}_c)_0$	t_{esc}	a	e	$\bar{\omega}$	t	l
−0.001	7.1	0.5507	0.4006	3.4387	15.9	6.0417
−0.002	6.728	0.5519	0.3993	3.1506	15.728	6.2402
−0.003	6.8	0.5523	0.3965	3.0200	15.6	6.1410
−0.005	6.706	0.5545	0.3917	2.9417	15.706	0.0656
−0.010	6.5	0.5620	0.3681	3.3652	16.3	0.3012
−0.011	6.7	0.5619	0.3683	3.5573	16.3	0.2043
−0.012	6.806	0.5516	0.3977	3.8534	15.806	5.7243
−0.013	13.8	0.5531	0.3963	4.0860	22.8	6.0847
−0.014		Impacts the lunar surface at $t = 8.630$				
−0.040	5.7	0.5539	0.3941	2.1517	15.1	0.3608
−0.050	5.3	0.5450	0.4218	1.5686	13.7	5.8786

The maximum of e_c then occurs for $t = 16.30$ with $a_c = 0.5620$, $e_c = 0.3681$; these compare with the previously cited values ($t = 16.25$, $a_c = 0.5621$, $e_c = 0.3681$) found using entirely different software and hardware, as discussed in Ref. 4. The only differences are attributable to the selection of time intervals for data printout (owing to the relative speeds of the CDC 3300 and the IBM 370-158 used in the two studies), so the results reported here may be regarded as fully reliable.

Table 4 shows the effect of $(\dot{x}_c)_0$ on this peak in e_c, the data being taken directly from the printout. To interpret these data, it first is necessary to consider the initial motion in the lunar vicinity; for $(\dot{x}_c)_0 = -0.01$ this is given in Fig. 5. Then in Table 4, t_{esc} is the escape time, or the time for motion from L2 to escape into Earth orbit with $r_{cm} > 0.2$. This motion involves a minimum r_{cm} close to 0.025, so that it is more or less fortuitous whether the time step $\Delta t = 0.0001$ is triggered.

For most of the cases of Table 4, the initial motion is very much like Fig. 5, and in particular it is seen that t_{esc} has only a slight dependence on $(\dot{x}_c)_0$. But for $(\dot{x}_c)_0 < -0.012$ this is not the case. This is explicable in terms of the theory of capture.[20] The motion of Fig. 5 is constrained by the zero-velocity curves of the restricted three-body problem. Near the Moon, these curves constitute a "bottle" with "bottlenecks" near L1 and L2, the L1 bottleneck being wider. For $(\dot{x}_c)_0 \leq -0.013$ the motion misses this bottleneck and returns to the interior of the bottle; escape occurs on a subsequent passage near L1. For $(\dot{x}_c)_0 \leq -0.014$, the motion impacts the lunar surface and only for $(\dot{x}_c)_0 \lesssim -0.03$ does the motion clear the Moon again and escape as before.

Hence, as a subsidiary result, it is seen that there exists a class of trajectories passing tangent to the lunar surface and passing L2 with relative velocity ~0.013, some 13 m/s. This follows from symmetry properties of orbits in the restricted three-body problem. Since this L2 velocity is an order of magnitude lower than that associated with the Moon-L2 trajectories of Refs. 4-6, it is of obvious interest whether such trajectories are of use. The answer appears to be no, because these trajectories display a highly complex looped structure and hence appear exorbitantly sensitive to small velocity errors at launch. Nevertheless, this question will be treated further in a subsequent communication.

Once the cargo has escaped past L1 and reached Earth orbit, its subsequent variation of a_c, e_c, $\bar{\omega}_c$ is exemplified by Fig. 6. These curves result from $(\dot{x}_c)_0 = -0.01$ and thus are equivalent to Fig. 1, to which Fig. 6 is keyed. It is clear that in the intervals between lunar encounters, the elements are rather well-behaved, at least in this restricted three-body approximation.

What is the fate of the cargo following such lunar encounters? This also would be, quite possibly, the fate of an L2 colony following failure of the stationkeeping systems. Table 5 describes some cases obtained by integration for selected values of $(\dot{x}_c)_0$. It is seen that while the cargo may suffer perturbations similar to those of Fig. 5, it will often be recaptured into lunar orbit. If it subsequently escapes through the L2 bottleneck, its behavior will resemble Fig. 5; but it may escape past L2. Then it is effectively lost, at least for a considerable time; and it may well be that more detailed computations would show cases where it is ejected into solar orbit. The possibility of collision with the Moon also exists. So as an operational mission rule, it appears desirable to require that the cargo be transferred to the colony prior to its first lunar encounter subsequent to escape, e.g. prior to apogee 7 in Fig. 1.

The near-invariance of $\bar{\omega}_c$ during this interval, together with the fact that $\bar{\omega}_c$ is almost entirely determined by the value of lunar mean longitude when the cargo leaves L2, then permits a cargo transfer strategy that, in principle, can always assure that ΔV_I is close to the value of Table 3 and that the terminal transfer is Hohmann. The strategy is as follows:

1) Depart L2 at a time so selected that when performing ΔV_I, $(\bar{\omega}_{cargo} - \bar{\omega}_{colony}) = 0$ or 180 deg.

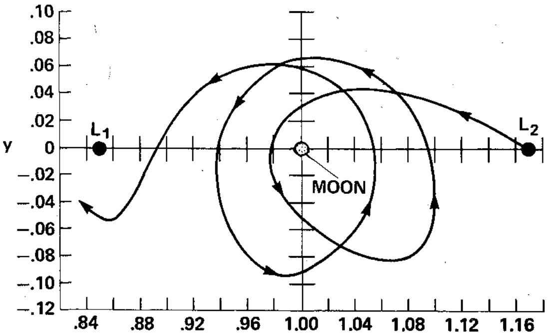

Fig. 5 Initial motion away from L2 and near the Moon, in the circular restricted three-body problem.

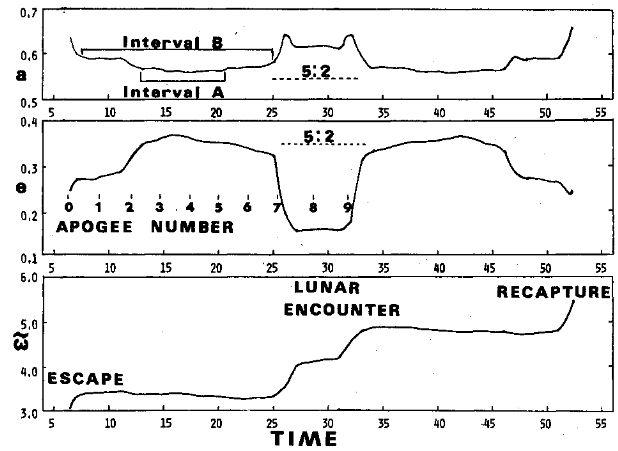

Fig. 6 Behavior of orbital elements for the trajectory of Fig. 1. During interval A, the motion is quite close to the nominal 5:2 resonant orbit of Fig. 2.

Table 5 Characteristics of certain transfer trajectories

$(\dot{x}_c)_0$	−0.001	−0.005	−0.01	−0.013	−0.05
Long-term fates, integrated thru $t \sim 50$	Lunar recapture at $t=25.3$; re-escape at $t=35.3$. $e_{max}=0.4109$ at $t=43.9$ with $\bar{\omega}=0.2624$, $l=5.952$. Nothing new to $t=51.5$.	Lunar recapture at $t=25.106$; re-escape at $t=30.438$ thru L2 with $a=3.0411$, $e=0.6026$. Nothing new to $t=52.038$.	Lunar close encounter near $t=25$; lunar recapture at $t=52.3$. Nothing new to $t=54.626$.	Lunar recapture at $t=32.4$; re-escape at $t=38.1$ thru L2 with $a=6.8023$, $e=0.8233$. At $t=49.3$, $a \rightarrow 4.4714$, $e=0.7420$	Lunar close encounters near $t=23$, $t=42$. No lunar recapture to end of integration at $t=59.9$.
Interval A:[a]					
Range of t	12.5-19.5	12.506-19.706	12.9-20.5	19.0-26.2	9.9-16.9
Range of a	0.5487-0.5551	0.5512-0.5576	0.5584-0.5650	0.5505-0.5579	0.5427-0.5504
Range of e	0.3773-0.4006	0.3684-0.3917	0.3456-0.3681	0.3684-0.3963	0.4008-0.4206
Δl	16.9899	17.3737	18.0197	17.3954	17.2786
$\Delta\theta_{2:1}$	2.8667	2.8470	2.6859	2.8687	3.1554
$\Delta\theta_{5:2}$	−0.4534	−0.5680	−0.9187	−0.5463	−0.1647
Interval B:[a]					
Range of t	7.9-23.9	7.306-24.106	7.5-24.9	14.6-31.2	5.9-22.3
Δl	37.8650	39.3777	40.1206	39.0851	38.7666
$\Delta\theta_{2:1}$	5.5834	5.4820	5.0144	5.5929	5.6780
$\Delta\theta_{5:2}$	−2.0054	−2.4862	−3.2385	−2.2804	−2.1006

[a] Note: $\theta_{2:1} = l - 2.0176\,t$, $\theta_{5:2} = l - 2.4919\,t$; numerical coefficients are derived from Table 1.

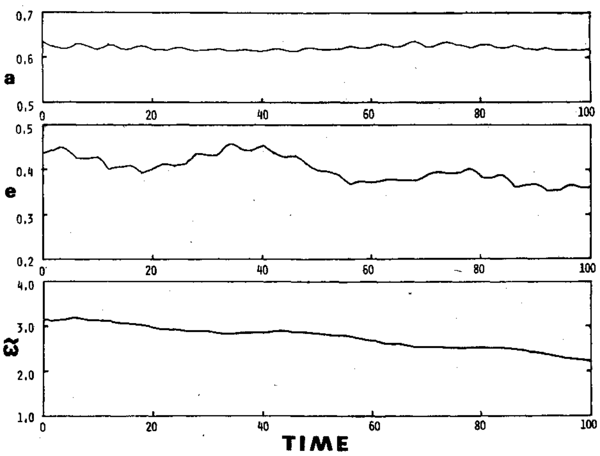

Fig. 7 Detail of the behavior of the orbital elements of the 2:1 resonant orbit of Fig. 4.

2) Escape to high Earth orbit and coast along in an orbit confocal with that of the colony until cargo and colony are about to reach apogee simultaneously, during the available coast time (prior to lunar encounter of the cargo).

3) Use $(\dot{x}_c)_0$ as a control to introduce minor variations in cargo motion such that steps 1 and 2 are achieved.

4) Conduct a Hohmann transfer to the colony.

Figure 6 shows two time intervals available for the coast maneuver (step 2). Interval B corresponds to that between apogee 1 and apogee 7. Interval A is approximately between apogees 2 and 5. Interval A is particularly desirable because e_c then is very close to the maximum of Table 4, the elements are quite stable, and the cargo motion is farthest removed from closely approaching the Moon. Table 5 then gives data on the change in the phase angle, $\Delta\theta = l_{cargo} - l_{colony}$, i.e., the difference in their mean anomalies. These data measure the ease with which step 2 can be achieved in practice. A value of π is required in order that step 2 can always be achieved.

It is seen that during interval A, the motion is very close to the nominal 5:2 resonant colony orbit. As a result, $\Delta\theta$ is small and it is usually a fortuitous coincidence if the cited transfer program can be carried out. But for the 2:1 resonance, $\Delta\theta$ nearly equals π and the transfer program can almost always be conducted. If one adopts the less restrictive interval B, then the transfer program still cannot always be conducted, with the colony at the 5:2 resonance. But if the colony is at the2:1 resonance, then the mission planner often has two opportunities for the transfer and can select the one with lower ΔV_l.

An additional feature of the 2:1 orbit is the possibility of departing L2 in such a manner as to reduce or eliminate the plane-change requirement of Table 2. This follows from the fact that the lunar-launched mass is not caught at L2 itself but rather is caught while executing a motion somewhat like a halo orbit, near L2.[4,6] Hence one may control the out-of-plane motion in such a manner as to produce a trajetory as in Fig. 5, post-L2-release, inclined by up to 6.9 deg to the lunar equator. Such an orbit still escapes the lunar vicinity as does that of Fig. 1,[20] but in Earth orbit the cargo has a value of $(i_c - i_m)$ of up to approximately 1 deg.

There remains for consideration the question of whether perturbations such as those in Fig. 4 might not increase ΔV_l markedly, even though the associated transfer is Hohmann. To gain insight into this question, Fig. 7 is an expanded plot of the data of Fig. 4 for the 2:1 resonance, plotted in the format of Fig. 6, but with the time scale compressed twofold in comparison to Fig. 6. It is seen that the colony semimajor axis and perigee regression rates are very stable, so there will be ample opportunity for carrying out step 2 successfully. On the other hand, the colony eccentricity deviates decidedly from its nominal value.

To estimate the associated increase in ΔV_l, the worst-case colony elements plotted are associated with the maximum deviation in e. In Fig. 7, this occurs at $t=92.9$ with $a=0.6243$, $e=0.3477$. For a coplanar transfer from ($a_c=0.5621$, $e_c=0.3681$), $\Delta V_l = 0.06739 = 68.95$ m/s. This compares with the nominal coplanar transfer associated with Table 3, 51.73 m/s.

Accordingly, these operational considerations show that the 2:1 resonance is advantageous, and it is recommended that it be regarded as the principal candidate for location of a space colony.

Conclusions

The present discussions are keyed to a specific scenario for space colonization, or for use of extraterrestrial resources in space industry. Under this scenario, 1) the source of resources is the Moon, 2) the launch mode is by mass-driver, 3) the catching-point is L2, and 4) the location for major industrial operations is the colony.

This scenario has received more attention than any other. But it is important to note that it may require revision. Easy access to asteroidal resources could impact points 1 and 4. Availability of lunar-derived chemical rocket propellants could impact 2. Improvements in mass-driver launch accuracy could impact 3 by rendering attractive direct launch of lunar material to L5 or L4, so that the mass-catcher might operate in the near vicinity of a colony.

However, accepting the above scenario leads to the conclusion that the optimal colony location is a 2:1 resonant orbit of Earth with elements $a=0.6342$ (Moon = 1), $e=0.4399$, $i=5.791$ deg, $\Omega=\Omega_{\mathrm{Moon}}$. Such a colony can always be reached from L2 via a transfer sequence involving a Hohmann transfer and with nominal $\Delta V=60$ to 80 m/s. These conclusions follow from realistic models that incorporate numerical four-body perturbations, three-dimensional effects including nodal regression under secular and commensurability-type resonances, and a nominal transfer trajectory studied in the restricted three-body problem, this trajectory taking account of lunar close approaches.

Acknowledgments

This work has benefited from the comments of Hans Scholl and Joachim Schubart. This work represents one phase of research conducted at the Max-Planck-Institut für Kernphysik, with support through a research fellowship with the Alexander von Humboldt Foundation, Bonn, West Germany. Certain computations were performed at the California Institute of Technology and were assisted by Carolyn Porco.

References

[1] Heppenheimer, T. A. and Hopkins, M. M., "Initial Space Colonization: Concepts and R&D Aims," *Astronautics & Aeronautics*, Vol. 14, March 1976, pp. 58-64, 72.

[2] Heppenheimer, T. A., *Colonies in Space*, Stackpole Books, Harrisburg, Pa., 1977.

[3] Johnson, R. D. and Holbrow, C. (eds.), *Space Settlements: A Design Study*, NASA SP-413, 1977.

[4] Heppenheimer, T. A. and Kaplan, D., "Guidance and Trajectory Considerations in Lunar Mass Transportation," *AIAA Journal*, Vol. 15, April 1977, pp. 518-525.

[5] Heppenheimer, T. A., "Achromatic Trajectories and Lunar Material Transport for Space Colonization," *Journal of Spacecraft and Rockets*, Vol. 15, May-June 1978, pp. 176-183.

[6] Heppenheimer, T. A., "A Mass-Catcher for Large-Scale Lunar Material Transport," *Journal of Spacecraft and Rockets*, Vol. 15, July-Aug. 1978, pp. 242-249.

[7] Heppenheimer, T. A., "Adiabatic Invariants and Phase Equilibria for First-Order Orbital Resonances," *Astronomical Journal*, Vol. 80, June 1975, pp. 465-472.

[8] Moulton, F. R., "On the Stability of Direct and Retrograde Satellite Orbits," *Monthly Notices of the Royal Astronomical Society*, Vol. 75, Dec. 1914, pp. 40-57.

[9] Heppenheimer, T. A., "On the Presumed Capture Origin of Jupiter's Outer Satellites," *Icarus*, Vol. 24, Feb. 1975, pp. 172-180.

[10] Brouwer, D. and Clemence, G. M., *Methods of Celestial Mechanics*. Academic Press, New York, 1961.

[11] O'Neill, G. K., "The Colonization of Space," *Physics Today*, Vol. 27, Sept. 1974, pp. 32-40.

[12] O'Neill, G. K., "Engineering a Space Manufacturing Center," *Astronautics & Aeronautics*, Vol. 14, Oct. 1976, pp. 20-28.

[13] Schutz, B. E., "Orbital Mechanics of Space Colonies at L4 and L5 of the Earth-Moon System," AIAA Paper 77-33, Aerospace Sciences Meeting, 1977.

[14] O'Leary, B. T., "Mining the Apollo and Amor Asteroids," *Science*, Vol. 197, July 22, 1977, pp. 363-366.

[15] "Final Report for Lunar Libration Point Flight Dynamics Study," General Electric Co., Contract NAS-5-11551, April 1969.

[16] Heppenheimer, T. A., "Transport of Lunar Material to the Sites of the Colonies," in *Space Manufacturing Facilities* (*Space Colonies*), edited by J. Grey, AIAA, New York, 1977, pp. 13-23.

[17] Farquhar, R. W. and Kamel, A. A., "Quasi-Periodic Orbits Around the Translunar Libration Point," AIAA Paper 72-935, Astrodynamics Specialists Conference, 1972.

[18] Breakwell, J. V. and Brown, J. V., "The 'Halo' Family of 3-Dimensional Periodic Orbits in the Restricted Three-Body Problem," AIAA Paper 76-825, Astrodynamics Specialists Conference, 1976.

[19] Scholl, H. and Froeschle, C., "Asteroidal Motion at the 5/2, 7/3 and 2/1 Resonances," *Astronomy and Astrophysics*, Vol. 42, 1975, pp. 457-463.

[20] Heppenheimer, T. A. and Porco, C., "New Contributions to the Problem of Capture," *Icarus*, Vol. 30, Feb. 1977, pp. 385-401.